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On the Convergence of Hakopian Interpolation and Cubature*

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In this paper, we improve a theorem on Hakopian interpolation due to the first author. We also propose a new kind of cubature scheme on the disk, which is convergent for the continuous functions. © 1997 Academic Press

1. INTRODUCTION

In 1982, Hakopian [1] proposed a new kind of multivariate interpolation, which is a development of Kergin interpolation (see [2–4]). Wang and Lai [5] studied in 1984 the remainder of the interpolation and established the convergence for the analytic functions. A Lagrange representation of the bivariate Hakopian interpolation was given in [6, 7]. We proved also a convergence theorem for sufficiently smooth functions defined on the unit disk (1986). Recently, in 1992, the convergence of the derivative of the bivariate Hakopian interpolation on the disk was also discussed in [8]. Our main purpose in this paper is to discuss further the convergence of Hakopian interpolation and to propose a new kind of cubature scheme based on this kind of interpolation.

Let D denote the unit disk

$$D = \{X = (x, y)^T \mid x^2 + y^2 \leqslant 1\},$$

and let $C(D)$ denote the space of all continuous functions defined on D. Set

$$E_m(f) := \inf_{p \in \pi_m(R^2)} \max_{X \in D} |f(X) - p(X)|,$$

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where $\pi_m(R^2)$ is the space of all real bivariate polynomials of total degree $\leqslant m$. Given the natural number $n \geqslant 2$, we choose

$$X^k = \left(\cos \frac{2k\pi}{n}, \sin \frac{2k\pi}{n} \right)^T, \qquad k = 1, 2, ..., n,$$

as the nodes of Hakopian interpolation on the disk. Denote

$$\int_{[X^k, X^l]} f = \int_0^1 f(X^k + t(X^l - X^k))\, dt.$$

On the basis of the results in [6], Liang has given the following theorems (see [7]).

THEOREM 1. *For any $f(X) \in C(R^2)$, there exists a unique polynomial $P(X) = H_n(f; X) \in \pi_{n-2}(R^2)$, such that*

$$\int_{[X^k, X^l]} f - P = 0, \qquad 0 \leqslant k < l \leqslant n.$$

Here $P(X) = H_n(f; X)$ denotes the Hakopian interpolation polynomial of $f(x, y) \in C(D)$ with respect to the n nodes, and we have the following representation:

$$H_n(f; X) = \sum_{1 \leqslant k < l \leqslant n} L_{k,l}(X) \cdot \int_{[X^k, X^l]} f,$$

where

$$L_{k,l}(X) = \frac{w'_{k,l}\left(r \cos \left(\frac{k+l}{n} \pi - \theta \right) \right)}{w'_{k,l}\left(\cos \frac{l-k}{n} \pi \right)},$$

$$w_{k,l}(t) = \prod_{j=1, j \neq k}^{n} \left(t - \cos \frac{k+l-2j}{n} \pi \right),$$

$$X = (x, y)^T = (r \cos \theta, r \sin \theta)^T.$$

THEOREM 2. *There exists an absolute constant A such that for any $f(X) \in C(D)$ and any $X \in D$, the following inequality is valid*:

$$|f(X) - H_n(f; X)| \leqslant \frac{A E_{n-2}(f)}{\sqrt{1 - X^T X}}\, n \log n.$$

A number of improvements on Theorem 2 are made in Section 2. In Section 3, after introducing a new kind of cubature scheme for the bivariate continuous functions defined on the disk, the convergence of integration of Hakopian interpolation is established and the algebraic exactness of the scheme is also discussed.

2. THE CONVERGENCE OF HAKOPIAN INTERPOLATION ON THE DISK

Let

$$T_m(t) = \cos(m \cdot \arccos t),$$

$$U_{m-1}(t) = T'_m(t),$$

$$t = r\cos\left(\frac{k+l}{n}\pi - \theta\right) = \cos\tilde{\theta}, \qquad \tilde{\theta} \in [0, \pi].$$

We need the following lemmas.

LEMMA 3. *If $n = 2m$, then we have*

$$\sum_{j=1}^{m} \sum_{i=1}^{m-1} \frac{\sin^2\theta_i \, |U_{m-1}(t)|}{m^2 \, |t - \cos\theta_i|} = O(m \log m), \tag{1}$$

where

$$t = r\cos\left(\frac{j}{m}\pi - \theta\right), \qquad j = 1, 2, ..., m;$$

$$\theta_i = \frac{i}{m}\pi, \qquad i = 1, ..., m-1.$$

$$\sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\sin^2\theta_i \, |U_{2m-1}(t)|}{m^2 \, |t - \cos\theta_i|} = O(m \log m), \tag{2}$$

where

$$t = r\cos\left(\frac{2j-1}{2m}\pi - \theta\right), \qquad j = 1, 2, ..., m;$$

$$\theta_i = \frac{2i-1}{2m}\pi, \qquad i = 1, ..., m.$$

Proof. We are going to prove (1) only. The proof of (2) is similar. In fact, setting $t=\cos\tilde{\theta}$, we have

$$\begin{aligned}\sum_{i=1}^{m-1}\frac{\sin^2\theta_i\,|U_{m-1}(t)|}{m^2\,|t-\cos\theta_i|}&=\sum_{i=1}^{m-1}\frac{\sin^2\theta_i\cdot m\cdot|\sin m\tilde{\theta}|}{m^2\,|\cos\tilde{\theta}-\cos\theta_i|\,\sqrt{1-t^2}}\\&\leqslant\sum_{i=1}^{m-1}\frac{|\sin\theta_i-\sin\tilde{\theta}|\,|\sin\theta_i|\,m\,|\sin m\tilde{\theta}|}{m^2\,|\cos\tilde{\theta}-\cos\theta_i|\,\sqrt{1-t^2}}\\&\quad+\sum_{i=1}^{m-1}\frac{|\sin\theta_i|\,m\,|\sin m\tilde{\theta}|}{m^2\,|\cos\tilde{\theta}-\cos\theta_i|}\\&=I_1+I_2.\end{aligned}$$

Noting that for $\theta_i, \tilde{\theta}\in[0,\pi]$,

$$\sin\theta_i>0,\qquad\sin\tilde{\theta}>0,$$

we have

$$\begin{aligned}I_1&\leqslant\sum_{i=1}^{m-1}\frac{|\sin\theta_i-\sin\tilde{\theta}|\,|\sin\theta_i+\sin\tilde{\theta}|\,m\,|\sin m\tilde{\theta}|}{m^2\,|\cos\tilde{\theta}-\cos\theta_i|\,\sqrt{1-t^2}}\\&=\sum_{i=1}^{m-1}\frac{2\cdot\left|\cos\frac{\tilde{\theta}+\theta_i}{2}\cdot\sin\frac{\tilde{\theta}-\theta_i}{2}\right|\cdot 2\left|\sin\frac{\tilde{\theta}+\theta_i}{2}\cdot\cos\frac{\tilde{\theta}-\theta_i}{2}\right|\,|\sin m\tilde{\theta}|}{m\cdot 2\left|\sin\frac{\tilde{\theta}+\theta_i}{2}\cdot\sin\frac{\tilde{\theta}-\theta_i}{2}\right|\sqrt{1-t^2}}\\&\leqslant\sum_{i=1}^{m-1}\frac{2\cdot|\sin m\tilde{\theta}|}{m\sqrt{1-t^2}}\leqslant\frac{2\cdot|\sin m\tilde{\theta}|}{\sqrt{1-t^2}}.\end{aligned}$$

we might as well assume that

$$\frac{k}{m}\pi<\theta<\frac{k+1}{m}\pi.$$

Since

$$r\leqslant 1\qquad\text{and}\qquad\left|\frac{\sin m\tilde{\theta}}{m\sqrt{1-t^2}}\right|=\left|\frac{\sin m\tilde{\theta}}{m\sin\tilde{\theta}}\right|\leqslant 1,$$

then

$$\frac{1}{2}\cdot\sum_{j=1}^{m}\frac{I_1}{m}\leqslant\sum_{j=1}^{m}\left|\frac{\sin m\tilde{\theta}}{m\sin\tilde{\theta}}\right|\leqslant 4+\sum_{j=1}^{k-2}\left|\frac{\sin m\tilde{\theta}}{m\sin\tilde{\theta}}\right|+\sum_{j=k+3}^{m}\left|\frac{\sin m\tilde{\theta}}{m\sin\tilde{\theta}}\right|$$

$$\leqslant 4+\sum_{j=1}^{k-2}\frac{1}{m\sqrt{1-r^2\cos^2\left(\frac{j}{m}\pi-\theta\right)}}$$

$$+\sum_{j=k+3}^{m}\frac{1}{m\sqrt{1-r^2\cos^2\left(\frac{j}{m}\pi-\theta\right)}}$$

$$\leqslant 4+2\sum_{j=2}^{m-2}\frac{1}{\left|m\sin\frac{j}{m}\pi\right|}=O(\log m).$$

Therefore

$$\sum_{j=1}^{m} I_1=O(m\log m).$$

Next, similar to the results in [7], we can prove

$$I_2=\sum_{i=1}^{m-1}\frac{|\sin\theta_i||\sin m\tilde{\theta}|}{m\,|\cos\tilde{\theta}-\cos\theta_i|}=O(\log m).$$

Hence

$$\sum_{j=1}^{m} I_2=O(m\log m).$$

This completes the proof of (1).

LEMMA 4. *Let*

$$\lambda_n(X)=\sum_{1\leqslant k<l\leqslant n}|L_{k,l}(X)|.$$

Then we have

$$\lambda_n(X)=O(n\log n).$$

Proof. (I) If n is even, let $n=2m$ and then $L_{k,l}(X)$ can be expressed as follows:

$$L_{k,l}(X)=\frac{\sin^2\frac{l-k}{n}\pi}{m^2\left(t-\cos\frac{l-k}{n}\pi\right)}\left\{\frac{1}{2}U_{2m-1}(t)-\frac{(T_m(t))^2}{t-\cos\frac{l-k}{n}\pi}\right\},$$

if $l-k$ is odd;

$$L_{k,l}(X)=-\frac{\sin^2\frac{l-k}{n}\pi\cdot U_{m-1}(t)}{m^2\left(t-\cos\frac{l-k}{n}\pi\right)}\left\{2T_m(t)+\frac{(1-t^2)\,U_{m-1}(t)}{m^2\left(t-\cos\frac{l-k}{n}\pi\right)}\right\},$$

if $l-k$ is even.

We have

$$\lambda_n(X)\leqslant\sum\nolimits_1+\sum\nolimits_2,$$

where

$$\sum\nolimits_1=\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{\sin^2\frac{2i-1}{2m}\pi}{m^2\left|t-\cos\frac{2i-1}{2m}\pi\right|}\left|\frac{1}{2}U_{2m-1}(t)-\frac{(T_m(t))^2}{t-\cos\frac{2i-1}{2m}\pi}\right|,$$

$$t=r\cos\left(\frac{2j-1}{2m}\pi-\theta\right),\qquad j=1,2,...,m;$$

$$\sum\nolimits_2=\sum_{j=1}^{m}\sum_{i=1}^{m-1}\frac{\sin^2\frac{i}{m}\pi\,|U_{m-1}(t)|}{m^2\left|t-\cos\frac{i}{m}\pi\right|}\left|2T_m(t)+\frac{(1-t^2)\,U_{m-1}(t)}{m^2\left(t-\cos\frac{i}{m}\pi\right)}\right|,$$

$$t=r\cos\left(\frac{j}{m}\pi-\theta\right),\qquad j=1,2,...,m.$$

Taking into account (1) and the relations

$$|T_m(t)|=O(1),\ \frac{(1-t^2)\,U_{m-1}(t)}{m^2\left|t-\cos\frac{i}{m}\pi\right|}=O(1),$$

we get

$$\sum_2 = O(m \log m).$$

By use of (2) and the relations (see [9])

$$\left|\frac{\sin\theta_i \cdot T_m(t)}{m(t-\cos\theta_i)}\right| = O(1),$$
$$\sum_{j=1}^{m}\sum_{i=1}^{m}\frac{|\sin\theta_i|\cdot T_m(t)}{m\,|t-\cos\theta_i|} = O(m\log m), \tag{3}$$

we have

$$\sum_1 = O(m \log m).$$

Therefore

$$\lambda_n(X) = O(m\log m) = O(n\log n).$$

(II) If n is odd, let $n = 2m-1$ and let

$$R_{m-1}(t) = \frac{T_m(t)+T_{m-1}(t)}{t+1} = \frac{\cos\left(m-\frac{1}{2}\right)\tilde{\theta}}{\cos\frac{1}{2}\tilde{\theta}},$$

$$S_{m-1}(t) = \frac{T_m(t)-T_{m-1}(t)}{t-1} = \frac{\sin\left(m-\frac{1}{2}\right)\tilde{\theta}}{\sin\frac{1}{2}\tilde{\theta}}.$$

Then $L_{k,l}(X)$ can be expressed as follows:

$$L_{k,l}(X) = \frac{2\left(\sin\frac{l-k}{2m-1}\pi\right)^2 R_{m-1}(t)}{(2m-1)^2\left(t-\cos\frac{l-k}{2m-1}\pi\right)}\left\{2(1+t)\,R'_{m-1}(t)\right.$$
$$\left. -\frac{\left(1+\cos\frac{l-k}{2m-1}\pi\right)R_{m-1}(t)}{t-\cos\frac{l-k}{2m-1}\pi}\right\}, \qquad \text{if} \quad l-k \text{ is odd;}$$

$$L_{k,l}(X)=\frac{2\left(\sin\dfrac{l-k}{2m-1}\pi\right)^2 S_{m-1}(t)}{(2m-1)^2\left(t-\cos\dfrac{l-k}{2m-1}\pi\right)}\left\{2(1-t)\,S'_{m-1}(t)\right.$$

$$\left.-\frac{\left(1-\cos\dfrac{l-k}{2m-1}\pi\right)S_{m-1}(t)}{t-\cos\dfrac{l-k}{2m-1}\pi}\right\}, \qquad \text{if} \quad l-k \text{ is even.}$$

We can prove that

$$\sum_{j=1}^{m}\sum_{i=1}^{m-1}\frac{2\sin^2\dfrac{l-k}{2m-1}\pi\,|R_{m-1}(t)|}{(2m-1)\left|t-\cos\dfrac{l-k}{2m-1}\pi\right|}=O(n\log n),$$

where

$$t=r\cos\left(\frac{2j+1}{2m-1}\pi-\theta\right), \qquad j=1,2,...,m;$$

$$l-k=2i-1, \qquad i=1,...,m-1.$$

Furthermore

$$\frac{(1+t)|R'_{m-1}|}{2m-1}=O(1),$$

$$\frac{\left(1+\cos\dfrac{l-k}{2m-1}\pi\right)|R_{m-1}(t)|}{(2m-1)\left|t-\cos\dfrac{l-k}{2m-1}\pi\right|}=O(1).$$

Similar relations hold also for even $l-k$. So we have

$$\lambda_n(X)=O(n\log n),$$

which completes the proof of Lemma 4.

The following theorem is our main result:

THEOREM 5. *There exists an absolute constant A such that for any $X\in D$, $f(X)\in C(D)$, and $n\geqslant 2$, we have*

$$|f(X)-H_n(f;X)|\leqslant AE_{n-2}(f)\cdot n\log n.$$

Proof. Let $Q_{n-2}(X) \in \pi_{n-2}(R^2)$ be the best approximation polynomial of $f(X)$ on D, i.e.,

$$\max_{X \in D} |f(X) - Q(X)| = E_{n-2}(f).$$

Then

$$|f(X) - H_n(f; X)| \leqslant |f(X) - Q_{n-2}(X)| + |H_n(f - Q_{n-2}; X)|$$
$$\leqslant (1 + \lambda_n(X)) \, E_{n-2}(f).$$

Now we apply Lemma 4 and complete the proof of the theorem.

From [8], we have

COROLLARY 6. *For any $f(X) \in C^k(D)$, $n \geqslant 2$ and any $X \in D$,*

$$|f(X) - H_n(f; X)| = O(n^{1-k} \log n).$$

3. THE CONVERGENCE OF INTEGRATION OF THE TWO-VARIABLE HAKOPIAN INTERPOLATION

We first give several auxiliary lemmas.

LEMMA 7. *Let*

$$A_{k,l} = \iint_D L_{k,l}(x, y) \, dx \, dy. \tag{4}$$

Then we have

$$A_{k,l} = \frac{4\pi}{n^2} \sin^2 \frac{l-k}{n} \pi.$$

Proof. For the sake of simplicity, we write

$$f(t) = w_{k,l}(t),$$

$$c = f'_t\left(\cos \frac{l-k}{n} \pi\right).$$

Using the rotation transformation

$$t = r\left(\cos\frac{k+l}{n}\pi - \theta\right) \xlongequal{\alpha = ((k+l)/n)\pi} r\cdot\cos\alpha\cdot\cos\theta + r\cdot\sin\alpha\cdot\sin\theta$$

$$= x\cdot\cos\alpha + y\cdot\sin\alpha,$$

$$t' = -x\cdot\sin\alpha + y\cdot\cos\alpha,$$

we get

$$A_{k,l} = \iint_D \frac{w'_{k,l}\left(r\cos\left(\frac{k+l}{n}\pi - \theta\right)\right)}{w'_{k,l}\left(\cos\frac{l-k}{n}\pi\right)}\,dx\,dy$$

$$= \frac{1}{c}\iint_D f'_t(\cdot)\,dx\,dy$$

$$= \frac{1}{c}\iint_D f'_t(\cdot)\,dt\,dt'$$

$$= \frac{1}{c}\int_{-1}^{1}\int_{-\sqrt{1-(t')^2}}^{\sqrt{1-(t')^2}} f'_t(\cdot)\,dt\,dt'$$

$$\xlongequal{t' = \sin\theta} \frac{1}{c}\int_{-\pi/2}^{\pi/2} (f(\cos\theta) - f(-\cos\theta))\cos\theta\,d\theta$$

$$= \frac{1}{c}\int_0^{2\pi} f(\cos\theta)\cos\theta\,d\theta.$$

Next, we prove the lemma in the following two cases:

(I) If n is even, let $n = 2m$. It is not difficult to verify that

$$f(t) = \begin{cases} c_0\dfrac{\cos^2(m\cdot\arccos t)}{t - \cos\dfrac{l-k}{n}\pi} & \text{if } \; l-k \text{ is odd;} \\[2ex] c_0\dfrac{\sin^2(m\cdot\arccos t)}{t - \cos\dfrac{l-k}{n}\pi} & \text{if } \; l-k \text{ is even,} \end{cases}$$

$$c = f'_t\left(\cos\frac{l-k}{n}\pi\right) = c_0\frac{m^2}{\sin^2\dfrac{l-k}{n}\pi},$$

where c_0 is a constant.

Assume first that $l-k$ is odd. Then

$$A_{k,l}=\frac{\sin^2\frac{l-k}{n}\pi}{m^2}\int_0^{2\pi}\frac{\cos^2 m\theta}{\cos\theta-\cos\frac{l-k}{n}\pi}\cos\theta\,d\theta$$

$$=\frac{\sin^2\frac{l-k}{n}\pi}{m^2}\left\{\int_0^{2\pi}\cos^2 m\theta\,d\theta+\cos\frac{l-k}{n}\pi\int_0^{2\pi}\frac{\cos^2 m\theta}{\cos\theta-\cos\frac{l-k}{n}\pi}\,d\theta\right\},$$

where

$$\int_0^{2\pi}\cos^2 m\theta\,d\theta=\int_0^{2\pi}\tfrac{1}{2}(1+\cos 2m\theta)\,d\theta=\pi.$$

Using the quadrature scheme (see [9])

$$\int_0^{2\pi}g(\theta)\,d\theta\cong\frac{\pi}{m}\sum_{i=1}^{2m}g(\theta_i),\qquad \theta_i=\frac{2i-1}{2m}\pi,\quad i=1,2,...,n$$

of trigonometric degree of precision $2m-1$ (where $g(\theta)$ is a trigonometric function with a period 2π), we have

$$\int_0^{2\pi}\frac{\cos^2 m\theta}{\cos\theta-\cos\frac{l-k}{n}\pi}\,d\theta=0$$

(we used here the L'Hospital's rule). Therefore

$$A_{k,l}=\frac{\sin^2\frac{l-k}{n}\pi}{m^2}\cdot\pi=\frac{4\pi}{n^2}\sin^2\frac{l-k}{n}\pi.$$

The deduction is similar in case of even $l-k$, where

$$g(\theta)=\frac{\sin^2 m\theta}{\cos\theta-\cos\frac{l-k}{n}\pi}.$$

The corresponding points here are

$$\theta_i=\frac{i}{m}\pi,\qquad i=1,...,n.$$

(II) If n is odd (say $n=2m-1$), we have

$$f(t)=\begin{cases} c_1 \dfrac{(\cos(m\cdot\arccos t)+\cos((m-1)\cdot\arccos t))^2}{(t+1)\left(t-\cos\dfrac{l-k}{n}\pi\right)} & \text{if} \quad l-k \text{ is odd;} \\ c_1 \dfrac{(\cos(m\cdot\arccos t)-\cos((m-1)\cdot\arccos t))^2}{(t-1)\left(t-\cos\dfrac{l-k}{n}\pi\right)} & \text{if} \quad l-k \text{ is even,} \end{cases}$$

$$c=f'_t\left(\cos\frac{l-k}{n}\pi\right)=c_1\frac{(2m-1)^2}{2\sin^2\dfrac{l-k}{n}\pi},$$

where c_1 is a constant.

In the following we deduce the same result in case of odd $l-k$. The even case is similar.

$$A_{k,l}=\frac{2\sin^2\dfrac{l-k}{n}\pi}{(2m-1)^2}\int_0^{2\pi}\frac{(\cos m\theta+\cos(m-1)\theta)^2}{\left(\cos\theta-\cos\dfrac{l-k}{n}\pi\right)(1+\cos\theta)}\cos\theta\,d\theta$$

$$=\frac{2\sin^2\dfrac{l-k}{n}\pi}{(2m-1)^2}\int_0^{2\pi}\frac{4\cos^2\dfrac{2m-1}{2}\theta\cos^2\dfrac{\theta}{2}}{\left(\cos\theta-\cos\dfrac{l-k}{n}\pi\right)2\cos^2\dfrac{\theta}{2}}\cos\theta\,d\theta$$

$$=\frac{4\sin^2\dfrac{l-k}{n}\pi}{(2m-1)^2}\int_0^{2\pi}\frac{\cos^2\dfrac{2m-1}{2}\theta\cdot\cos\theta}{\cos\theta-\cos\dfrac{l-k}{n}\pi}\,d\theta.$$

The rest deduction is similar to the case of even n, we choose here

$$g(\theta)=\frac{\cos^2\dfrac{2m-1}{2}\theta}{\cos\theta-\cos\dfrac{l-k}{n}\pi},\qquad \theta_i=\frac{2i-1}{2m-1}\pi,\quad i=1,2,...,2m-1.$$

So, at all events, we get

$$A_{k,l} = \frac{4\pi}{n^2} \sin^2 \frac{l-k}{n} \pi,$$

which completes the proof.

The lemma implies

$$\begin{aligned}
\iint_D H_n(f; x, y)\, dx\, dy &= \iint_D \left(\sum_{1 \leqslant k < l \leqslant n} L_{k,l}(x, y) \int_{[X^k, X^l]} f \right) dx\, dy \\
&= \sum_{1 \leqslant k < l \leqslant n} \int_{[X^k, X^l]} f \left(\iint_D H_n(f; x, y)\, dx\, dy \right) \\
&= \sum_{1 \leqslant k < l \leqslant n} A_{k,l} \int_{[X^k, X^l]} f \\
&= \sum_{1 \leqslant k < l \leqslant n} \frac{4\pi}{n^2} \sin^2 \frac{l-k}{n} \pi \int_{[X^k, X^l]} f.
\end{aligned}$$

In this way, we get the cubature scheme on the unit disk: $\forall f(x, y) \in C(D)$,

$$\iint_D f(x, y)\, dx\, dy \cong \sum_{1 \leqslant k < l \leqslant n} \frac{4\pi}{n^2} \sin^2 \frac{l-k}{n} \pi \int_{[X^k, X^l]} f. \qquad (*)$$

The geometrical interpretation of the coefficient $A_{k,l}$ is illustrated in Fig. 1. Let L denote the length of the chord between point 0 and point $(2(l-k)/n)\pi$ on the unit circle. Then

$$A_{k,l} = L^2 \cdot \frac{\pi}{n^2}.$$

LEMMA 8. *If $A_{k,l}$ is defined as in (4), then we have*

$$\sum_{1 \leqslant k < l \leqslant n} A_{k,l} = \pi.$$

Proof. By Theorem 1, the relation $(*)$ holds for all bivariate polynomials of degree $\leqslant n-2$. Since $n \geqslant 2$, the relation holds in particular for the constant function, which implies

$$\sum_{1 \leqslant k < l \leqslant n} A_{k,l} = \sum_{1 \leqslant k < l \leqslant n} \frac{4\pi}{n^2} \sin^2 \frac{l-k}{n} \pi = \pi.$$

This completes the proof.

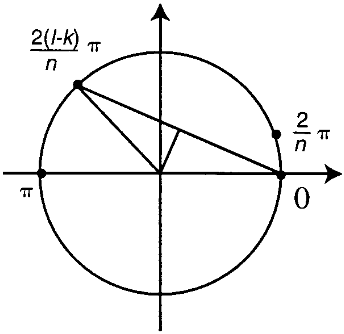

FIGURE 1

The following two theorems are the main results in this section.

THEOREM 9. *For a given $f(x, y) \in C(D)$, we have*

$$\lim_{n \to \infty} \iint_D H_n(f; x, y)\, dx\, dy = \iint_D f\, dx\, dy.$$

Proof. Let $P_{n-2}(x, y) \in \pi_{n-2}(R^2)$ be the best approximation polynomial of $f(x, y)$, i.e.

$$\max_{(x, y) \in D} |f(x, y) - P_{n-2}(x, y)| = E_{n-2}(f).$$

By Theorem 1 and Lemma 8, we have

$$\begin{aligned}
&\left| \iint_D H_n(f; x, y)\, dx\, dy - \iint_D f(x, y)\, dx\, dy \right| \\
&\leqslant \left| \iint_D H_n(f - P_{n-2}; x, y)\, dx\, dy \right| + \left| \iint_D (f - P_{n-2})\, dx\, dy \right| \\
&= \left| \sum_{1 \leqslant k < l \leqslant n} A_{k,l} \int_{[X^k, X^l]} (f - P_{n-2}) \right| + \left| \iint_D (f - P_{n-2})\, dx\, dy \right| \\
&\leqslant \sum_{1 \leqslant k < l \leqslant n} A_{k,l} \left| \int_{[X^k, X^l]} (f - P_{n-2}) \right| + \pi \cdot E_{n-2}(f) \\
&\leqslant \pi \cdot E_{n-2}(f) + \pi \cdot E_{n-2}(f) \\
&= 2\pi \cdot E_{n-2}(f) \to 0 \qquad \text{if} \quad n \to \infty.
\end{aligned}$$

This completes the proof on convergence of integration of Hakopian interpolation.

THEOREM 10. *The cubature scheme*

$$\iint_D f(x, y)\, dx\, dy \cong \sum_{1 \leqslant k < l \leqslant n} \frac{4\pi}{n^2} \sin^2 \frac{l-k}{n} \pi \cdot \int_{[X^k, X^l]} f \qquad (*)$$

is exact for any bivariate polynomial of degree $\leqslant n-1$ *at least.*

Proof. Noticing that by Theorem 1, the algebraic exactness of Hakopian interpolation is $n-2$ at least, then, in order to show that the theorem is true, we only need to prove the relation $(*)$ holds for all bivariate homogeneous polynomials of degree $n-1$. In order to do this, we choose

$$\alpha_j = \frac{2j\pi}{n}, \qquad j = 0, 1, ..., n-1,$$

and consider n homogeneous polynomials

$$f_j(x, y) = (x \cdot \cos \alpha_j + y \cdot \sin \alpha_j)^{n-1}, \qquad j = 0, 1, ..., n-1,$$

which are linearly independent from each other. They constitute a basis of all homogeneous polynomials of degree $n-1$. So we only need to check the equalities

$$\iint_D f_j(x, y)\, dx\, dy = \sum_{1 \leqslant k < l \leqslant n} \frac{4\pi}{n^2} \sin^2 \frac{l-k}{n} \pi \int_{[X^k, X^l]} f_j, \qquad j = 0, 1, ..., n-1, \tag{5}$$

where

$$\begin{aligned}
\int_{[X^k, X^l]} f_j &= \int_0^1 f_j\left((1-t) \cos \frac{2k\pi}{n} + t \cdot \cos \frac{2l\pi}{n}, (1-t) \sin \frac{2k\pi}{n} + t \cdot \sin \frac{2l\pi}{n}\right) dt \\
&= \int_0^1 \left\{(1-t) \cos\left(\frac{2k\pi}{n} - \alpha_j\right) + t \cdot \cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right\}^{n-1} dt \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^i \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{n-1-i}.
\end{aligned}$$

The right-hand side of (5) is

$$
\begin{aligned}
S :=& \sum_{1 \leqslant k < l \leqslant n} \frac{4\pi}{n^2} \sin^2 \frac{l-k}{n} \pi \int_{[X^k, X^l]} f_j \\
=& \frac{\pi}{n^2} \sum_{l=1}^{n} \sum_{k=1}^{n} \left(1 - \cos \frac{2(l-k)}{n} \pi\right) \int_{[X^k, X^l]} f_j \\
=& \frac{\pi}{n^3} \sum_{i=0}^{n-1} \sum_{l=1}^{n} \sum_{k=1}^{n} \left\{1 - \cos\left(\frac{2k\pi}{n} - \alpha_j\right) \cdot \cos\left(\frac{2l\pi}{n} - \alpha_j\right) \cdot \sin\left(\frac{2l\pi}{n} - \alpha_j\right)\right\} \\
& \times \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^i \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{n-1-i} \\
=& \frac{\pi}{n^3} \sum_{i=0}^{n-1} \left\{ \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^i \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{n-1-i} \right. \\
& - \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^{i+1} \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{n-i} \\
& - \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\sin\left(\frac{2k\pi}{n} - \alpha_j\right)\right)\left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^i \\
& \left. \times \sin\left(\frac{2l\pi}{n} - \alpha_j\right)\left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{n-1-i} \right\}.
\end{aligned}
$$

We assert that if $i = 0, n-1$, the later two terms in the sum above are zero. Since the trigonometric exactness of the quadrature scheme

$$\int_0^{2\pi} T(\theta)\, d\theta = \frac{2\pi}{n} \sum_{k=1}^{n} T\left(\frac{2(k-j)}{n} \pi\right) \tag{6}$$

(where j is a constant) is $n-1$, it follows that

$$\sum_{k=1}^{s} \cos\left(\frac{2k\pi}{n} - \alpha_j\right) = \frac{n}{2\pi} \int_0^{2\pi} \cos\theta\, d\theta = 0. \tag{7}$$

Therefore, in case $i = 0$, the second term in the sum is

$$\sum_{k=1}^{n} \cos\left(\frac{2k\pi}{n} - \alpha_j\right) \cdot \sum_{i=1}^{n} \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^n = 0.$$

The rest of the proof of the assertion is similar. Hence

$$S = \frac{\pi}{n^3} \left\{ \sum_{i=0}^{n-1} \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^i \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{n-1-i} \right.$$
$$- \sum_{i=1}^{n-2} \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^{i+1} \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{n-i}$$
$$- \sum_{i=1}^{n-2} \sum_{l=1}^{n} \sum_{k=1}^{n} \sin\left(\frac{2k\pi}{n} - \alpha_j\right)\left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^i$$
$$\left. \times \sin\left(\frac{2l\pi}{n} - \alpha_j\right) \cdot \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{n-i-1} \right\}.$$

Next, we prove the theorem in the following two cases:

(I) If n is even, using the fact that

$$\int_0^{2\pi} \cos^m \theta \, d\theta = 0, \qquad \text{if } m \text{ is odd}, \tag{8}$$

we conclude that the left hand side of (5) is

$$\iint_D f_j \, dx\, dy = \int_0^1 r^n \, dr \int_0^{2n} \cos^{n-1}(\theta - \alpha_j) \, d\theta = 0.$$

On the other hand, since the sum of the exponents of $\cos((2l\pi/n) - \alpha)$ and $\cos((2k\pi/n) - \alpha_j)$ is odd, there must be one term of S with odd exponent which is no greater than $n-1$. By (7) and quardrature scheme (6), we can get $S = 0$, which shows that the relation (5) is true in this case.

(II) We require the following lemma in case of odd n.

LEMMA 11. *For a nonnegative integer n, the following relation holds*:

$$J_m \triangleq \sum_{i=0}^{m} \frac{(2i-1)!!\,(2m-2i-1)!!}{(2i+2)!!} (2m-2i)!! \equiv \frac{(2m+1)!!}{(2m+2)!!}. \tag{9}$$

The proof is omitted.

Let $n = 2m+1$, using the integral formula:

$$\int_0^{2\pi} \cos^{2m}\theta \, d\theta = \frac{(2m-1)!!}{(2m)!!} \cdot 2\pi, \tag{10}$$

we calculate the left hand side of (5):

$$\iint_D f_j(x, y)\, dx\, dy = \int_0^1 dr \int_0^{2\pi} r^{2m+1} \cos^{2m}(\theta - \alpha_j)\, d\theta = \frac{(2m-1)!!}{(2m+2)!!} \cdot 2\pi.$$

On the other side, by (6), (8), (10) and Lemma 11, we get in the right hand side of (5):

$$\begin{aligned} S = {} & \frac{\pi}{(2m+1)\, n^2} \left\{ \sum_{i=0}^{m} \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^{2i} \left(\cos\left(\frac{2l\pi}{n} - \alpha_j\right)\right)^{2m-2i} \right. \\ & \left. - \sum_{i=0}^{m-1} \sum_{l=1}^{n} \sum_{k=1}^{n} \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^{2i+2} \left(\cos\left(\frac{2k\pi}{n} - \alpha_j\right)\right)^{2m-2i} \right\} \\ = {} & \frac{\pi}{(2m+1)\, n^2} \left\{ \sum_{i=0}^{m} \left(n^2 \cdot \frac{(2i-1)!!\,(2m-2i-1)!!}{(2i)!!\,(2m-2i)!!} \right.\right. \\ & \left.\left. - n^2 \frac{(2i+1)!!\,(2m-2i-1)!!}{(2i+2)!!\,(2m-2i)!!} \right) + n^2 \frac{(2m+1)!!}{(2m+2)!!} \right\} \\ = {} & \frac{\pi}{(2m+1)} \left\{ \sum_{i=0}^{m} \frac{(2i-1)!!\,(2m-2i-1)!!}{(2m+2)!!\,(2m-2i)!!} + \frac{(2m+1)!!}{(2m+2)!!} \right\} \\ = {} & \frac{\pi}{2m+1} \cdot 2 \frac{(2m+1)!!}{(2m+2)!!} = \frac{(2m-1)!!}{(2m+2)!!} \cdot 2\pi. \end{aligned}$$

The relation (5) is true in case of odd n.

To sum up, the cubature scheme $(*)$ is exact for all polynomials of degree $\leqslant n-1$, i.e., its algebraic degree of precision is at least $n-1$. The proof of Theorem 10 is finished.

In some simple cases as $n = 2$, we can prove that the algebraic degree of precision of the cubature scheme $(*)$ is just $n-1$.

4. GEOMETRICAL AND PHYSICAL INTERPRETATION

The value $\int_{[X^k, X^l]}$ can be regarded as observations on x-rays in medical diagnosis. So Hakopian interpolation will be of value in this field (for example, in computed tomography).

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